Efficient Valuation of Callable Bonds: the Dynamic Chebyshev Method

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Abstract

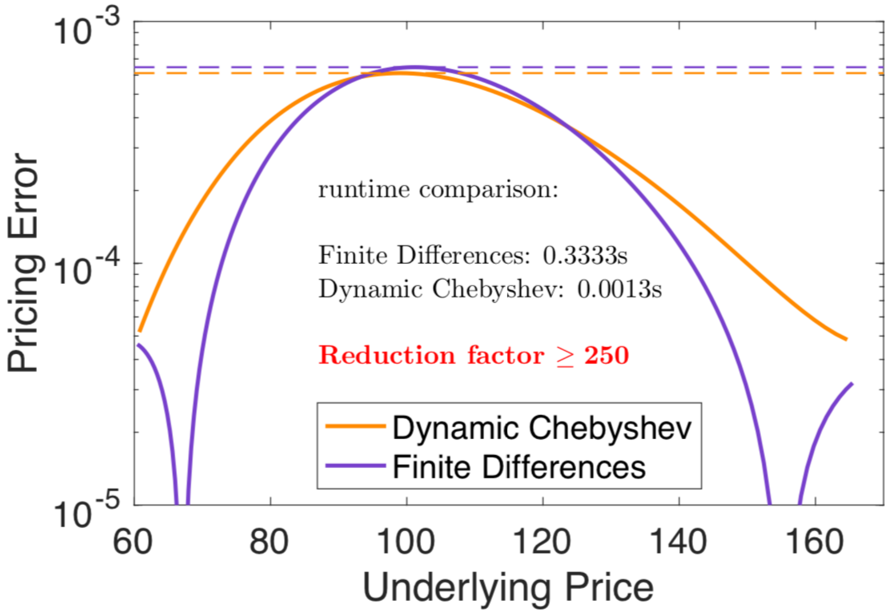
We investigate the application of the dynamic Chebyshev method to the pricing of callable bonds. Callable corporate bonds belong to the most important fixed income instruments in financial markets. Acknowledging the fact that the academic literature on efficient pricing of these instruments has been less prioritized in the past, we aim to close this gap. In order to put the method under test in a model that is relevant for practice, we consider a two-factor rate/credit model. We formulate the pricing problem as a dynamic programming problem and present a dynamic Chebyshev algorithm for this problem in a simplified setting. Finally, we present some numerical results.

1 Introduction

Callable corporate bonds belong to the most important fixed income instruments in financial markets. In the United States alone, over 70% of the 1.4 trillion USD of corporate bond issuance in 2019 were callable bonds.[1] Yet, a relatively small portion of the academic literature considers efficient pricing and calibration of these instruments in hybrid models that allow for stochastic and dependent interest and hazard rates.

In order to close this gap, we consider a new pricing approach called Dynamic Chebyshev from Glau et al. (2019) for pricing Bermudan type options. This has proven substantial efficiency gain in different settings, see also Glau et al. (2020). In order to get a first impresssion of the efficiency gain, we first consider a simplified problem. In the Black-Scholes model, we compare a Crank Nicolson Finite Difference solver (FD) with the Dynamic Chebyshev algorithm (DC). We consider a call option with maturity of one year and a strike price of 100. The DC algorithm runs with a weekly time stepping, i.e. with 52 time steps per year, we implemented both methods in MATLAB for a fair comparison. The parameters of the FD solver are chosen such that we obtain an accuracy comparable to the DC algorithm. We compute the errors by comparing the FD and the DC method to the explicit solution. The considerable efficiency gain of DC is seen in the figure. To be more precise, the errors for a range

[1]Source: Securities Industry and Financial Markets Association, research report "US corporate bond issuance" from March 9, 2020.



of initial asset prices and the efficiency gain of the DC algorithm is seen in the figure. We see that the same accuracy can be achieved with only a small fraction of the runtime.

Motivated by these impressive results, we exploit this highly efficient combination of dynamic programming with Chebyshev interpolation in order to price callable bonds. The approximation is fast and the accuracy can be swiftly increased to the requested level, and it also can be easily adapted to the desired model sophistication.

For all details we refer to Pötz (2020) and Glau et al. (2021).

2 Problem formulation

We consider defaultable corporate bonds that pay a fixed coupon rate. For such a (callable) bond with maturity T we introduce the following notation

- coupon payment dates $0 = T_0 < T_1 < \ldots < T_{n_C} = T$,
- coupon payments $C_i = cN(T_i - T_{i-1})$, $i = 1, \ldots, n_C$, where c is the coupon rate and N is the notional of the bond,
- exercise dates $t_1 < \ldots < t_m = T$ and strike prices $K(\tau)$.

We consider pricing of such callable bond in the presence of credit risk in a two-factor rate/credit model.

2.1 Two-factor model

We consider a two-factor hybrid model for the interest rate and the hazard rate of the credit intensity. The short rate $r(t)$ is modelled by a Hull-White model (see

Hull and White (1990)) and the hazard rate $\lambda(t)$ follows the model of Black and Karasinski (1991). More precisely, we write

$$r(t) = r(t, x_r(t)) = \phi_r(t) + x_r(t)$$
$$\lambda(t) = \lambda(t, x_\lambda(t)) = e^{\phi_\lambda(t)+x_\lambda(t)}$$

with the deterministic functions ϕ_r, ϕ_λ and the bivariate stochastic process $\mathbf{x}(t) := (x_r(t), x_\lambda(t))$ modelled by

$$\begin{aligned} \mathrm{d}x_\alpha(t) &= -a_r x_\alpha(t)\mathrm{d}t + \sigma_\alpha \mathrm{d}W_t^\alpha, \quad x_\alpha(0) = 0 \quad \text{for } \alpha = r, \lambda, \\ \mathrm{d}W_t^r \mathrm{d}W_t^\lambda &= \varrho \mathrm{d}t. \end{aligned} \tag{2.1}$$

The functions ϕ_r, ϕ_λ allow the model to be fitted to the term structure implied by the market.

We present our new methodology for a simplified problem in order to keep notation to a minimum.

2.2 Simplified pricing problem

In order to keep the presentation simple, we consider zero coupon bonds. The value of the callable zero coupon bond with no recovery is

$$V(0) = \min_{\tau \in \mathcal{T}_{Ex}} \mathbb{E}\big[Z(0,\tau) \min\{K(\tau), V^{Bond}(\tau)\}\big] \tag{2.2}$$

with the fair price $V^{Bond}(\tau)$ of a non-callable (but defaultable) bond at time τ, and risky discount factor

$$D^{risky}(s,t) = \mathbb{E}\big[Z(s,t)\big|\mathcal{F}_s\big] \quad \text{for} \quad Z(s,t) := \exp\Big(-\int_s^t \big(r(u) + \lambda(u)\big)\mathrm{d}u\Big),$$

and where $\mathcal{T}_{Ex}$ is the set of all stopping times which take values in the discrete set of exercise dates $\{t_1, \ldots, t_m\}$.

Before reformulating this as dynamic programming problem, we introduce the simplifications for the further presentation. We consider interest and hazard rate to be uncorrelated, $\varrho = 0$, and we restrict ourselves to bonds without coupon payments, $C_i = 0$.

Then reformulating the optimal stopping problem (2.2) as a dynamic programming problem yields

$$V_T(\mathbf{x}) = V_T^{Bond}(\mathbf{x})$$
$$V_{t_u}(\mathbf{x}) = \min\big\{K(t_u), \mathbb{E}_{t_u,\mathbf{x}}\big[Z(t_u, t_{u+1})V_{t_{u+1}}(\mathbf{x}(t_{u+1})\big]\big\}$$

where $\mathbb{E}_{t_u,\mathbf{x}}[\,\cdot\,]$ refers to the conditional expectation $\mathbb{E}[\,\cdot\,|\mathbf{x}(t_u) = \mathbf{x}]$ with $\mathbf{x} = (x_r, x_\lambda)$. Here, we assume the time points t_u form an equidistant grid and coincide with the exercise date.[2] We denote $\Delta t = t_{u+1} - t_u$.

[2]This can be relaxed, while all exercise dates need to be in the time grid, we may choose a finer time grid, the grid needs not to be equidistant.

First, conditioned on $\mathbf{x}(t_u) = \mathbf{x}$, we may approximate the factor

$$Z(t_u, t_{u+1}) = \exp\big(- \int_{t_u}^{t_{u+1}} (r(s) + \lambda(s))\mathrm{d}s\big) \approx \mathrm{e}^{-\Delta t(r(t_u,x_r)+\lambda(t_u,x_\lambda))} =: D^{risky}_{t_u,\mathbf{x}}$$

and obtain the following dynamic programming problem

$$\begin{aligned} V_T(\mathbf{x}) &= V_T^{Bond}(\mathbf{x}) \\ V_{t_u}(\mathbf{x}) &= \min\big\{K(t_u), D^{risky}_{t_u,\mathbf{x}} \mathbb{E}_{t_u,\mathbf{x}}[V_{t_{u+1}}(\mathbf{x}(t_{u+1}))]\big\}. \end{aligned} \tag{2.3}$$

3 Core idea of the new approach

We apply the Dynamyic Chebyshev method proposed in Glau et al. (2019). To this end, we approximate the value function $V_{t_u}(\mathbf{x})$ of a callable bond with a bivariate polynomial interpolation in every time step t_u. Assume we have an approximation of $V_{t_{u+1}} \approx \sum_{\mathbf{j}} c_{\mathbf{j}}^{u+1} p_{\mathbf{j}}$, for multi-index $\mathbf{j} = (j_r, j_\lambda)$, coefficient $c_{\mathbf{j}}^{u+1}$ and bivariate polynomials $p_{\mathbf{j}}$. At time t_u, we need to compute the function values on a grid of nodal points $\mathbf{x}_{\mathbf{k}} = (x_{k_r}, x_{k_\lambda})$ in order to approximate V_{t_u}, i.e.

$$\widehat{V}_{t_u}(\mathbf{k}) := \min\Big\{K(t_u), D_{t_u,\mathbf{x}_{\mathbf{k}}} \sum\nolimits_{\mathbf{j}} c_{\mathbf{j}}^{u+1} \mathbb{E}_{t_u,\mathbf{x}_{\mathbf{k}}}\big[p_{\mathbf{j}}(\mathbf{x}(t_{u+1}))\big]\Big\},$$

$\mathbf{k} = (k_r, k_\lambda)$. The model information is represented in the conditional expectations of the polynomials,

$$\Gamma^u_{\mathbf{j},\mathbf{k}} := \mathbb{E}_{t_u,x_{k_r},x_{k_\lambda}}\big[p_{\mathbf{j}}(x_r(t_{u+1}), x_\lambda(t_{u+1}))\big] = \mathbb{E}_{t_u,x_{k_r}}[p_{j_r}(x_r(t_{u+1}))]\mathbb{E}_{t_u,x_{k_\lambda}}[p_{j_\lambda}(x_\lambda(t_{u+1}))].$$

As polynomial basis, we choose a bivariate Chebyshev polynomials, for which the expectations $\Gamma^u_{\mathbf{j},\mathbf{k}}$ are swiftly computed from an iterative procedure.

The core model problem is easily adapted to more realistic models and more complex instruments, for which the method proves highly efficient as well. We in particular have implemented and tested the method for more complex settings:

Further option features

- **Risky callable bonds (with coupon payments)✔**
- **Recovery rate✔**
- **Credit default swaps ✔**

Further model complexity

- **Dependence of interest rate and hazard rate ✔**

4 Advantages

The scope of the method is for models up to three or four risk factors, there the method enjoys several key advantages:

- it delivers complete price function as in PDE approach
  $\Longrightarrow$ efficient exposure calculations

- compared to PDE approach: No matrix inversion required
  $\Longrightarrow$ higher efficiency
- swift computation of Greeks
  $\Longrightarrow$ Greeks and value function in one go
- offline online decomposition
  $\Longrightarrow$ even further efficiency gain possible
- modularity
- easy to implement and maintain
- easy to generalize to include further option and model features.

5 Some numerical results

The following numerical results are an excerpt of Pötz (2020), where the procedure is first run for different scenarios, with and $N = 128$ number of Chebyshev nodes in each of the two dimensions and 80 time steps per year to obtain a reference pricer. The related prices are shown in the first table. We show here results for three scenarios and a single choice of a DC method with lower accuracy.

Scenario	Correlation	Coupon	Recovery	Non-callable	Callable
1	Zero	No	No	66.8558	56.9431
2	Positive	No	No	67.2185	56.3092
3	Zero	Yes	No	87.1385	81.2028

The second table displays the relative pricing error of the dynamic Chebyshev method with $N = 32$ and 20, and with 40 time steps per year. We observe that the dynamic Chebyshev method with these parameters is for all scenarios able to achieve a relative errors below $5 * 10^{-4}$ and thus an error of less than 0.05%.

		1	2	3
$DC_{32,20}$	non-callable	$0.55 * 10^{-4}$	$1.09 * 10^{-4}$	$0.44 * 10^{-4}$
	callable	$0.21 * 10^{-4}$	$4.20 * 10^{-4}$	$1.84 * 10^{-4}$

In all scenarios, the runtime of the pricer was smaller or equal $6ms$, measured on a standard laptop using *Python*. The whole procedure, including calibration took $20ms$.

For a comprehensive study, we again refer to Pötz (2020), and for comparisons with benchmark methods we refer to Glau et al. (2021).

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